On Leibniz’ Characteristic Numbers

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ABSTRACT: In 1679, Leibniz wrote nine manuscripts on three different arithmetical models of Aristotelian logic. This was a part of his project of a “calculus universalis”. First we show the precise relations of these three models to each other by presenting three criteria which serve the purpose of classifying models of Aristotelian logic. This facilitates the understanding of Leibniz’ constructions. Our method is of special value for the sophisticated third model, the domain of which consists of pairs of natural numbers. We present a simple approach to Leibniz’ definitions which on first sight appear complicated. We show that it is possible to deduce, from the “universal positive” relation **a (=** “All ...”), the other three “Aristotelian relations” **i, o,** and **e**. – It has always been difficult to understand the exact nature of Leibniz’ characteristic numbers because of his misleading nomenclature, since he utilized the signs + and – in order to designate the “positive” and “negative” part of a characteristic number pair. We present a new interpretation of Leibniz’ symbolism, showing that the number pairs should be interpreted as numerator and denominator of a rational number. Thus we can identify the last model as the natural extension of the first and second one, showing the continuity in Leibniz’ different attempts towards an arithmetization of logic. – We close our paper by discussing two well known problematic aspects of Leibniz’ characteristic numbers, formulating two open questions concerning the formal structure of the system.

ZUSAMMENFASSUNG: Im Rahmen seines Projektes eines „calculus universalis“ entwarf Leibniz Anfang 1679 in neun Texten drei unterschiedliche Modelle der aristotelischen Logik mit Hilfe von Zahlen. Durch diese von ihm erfundenen „charakteristischen Zahlen“ wollte Leibniz die Schlussweisen der aristotelischen Logik auf rein arithmetische Rechnungen reduzieren. In der vorliegenden Arbeit zeigen wir genau, wie die drei Modelle untereinander zusammenhängen bzw. aufeinander aufbauen. Zu diesem Zweck geben wir drei Kriterien an, mit Hilfe derer sich Modelle der aristotelischen Logik klassifizieren lassen. Zum einen können wir dadurch die Leibnizschen Definitionen leicht nachvollziehen sowie auch die Stärken und Schwächen der einzelnen Modelle präzise beschreiben. Besonders für das ausgefeilte letzte Modell, dessen Grundbereich aus Paaren natürlicher Zahlen besteht, liefert unsere Methode einen ganz natürlichen Zugang zu den Leibnizschen Definitionen, die im Original recht sperrig wirken. Wir zeigen insbesondere – was Leibniz nicht herausgestellt hat -, dass allein aus der universell-positiven alle anderen drei „aristotelischen Relationen“ herleitbar sind. Ein Grund für manche Schwierigkeiten mit dem Verständnis der Leibnizschen charakteristischen Zahlen ist die von ihm verwendete Nomenklatur. Der problematischen Schreibweise mit Vorzeichen + und -, die in der Vergangenheit manchen Kommentator auf die falsche Fährte gelockt hat, geben wir eine neue Deutung: Dadurch entpuppt sich das letzte Modell als ganz natürliche Erweiterung des vorangehenden vom Grundbereich der natürlichen in den Bereich der positiven rationalen Zahlen. – Mit der vorliegenden Arbeit hoffen wir, der Verwendung charakteristischer Zahlen als theoretisches Instrument für die Untersuchung der aristotelischen Logik einen neuen Impuls geben zu können. Deshalb diskutieren wir am Schluss der Arbeit noch die aus der Literatur bekannten Stärken und Schwächen des Leibnizschen Modells und formulieren zwei offene Fragen zu diesem Komplex.

Introduction

In a small number of unpublished manuscripts[1] from the spring of the year 1679, Leibniz invented a new method of arithmetization of Aristotelian logic. These texts belong to his gigantic project of a calculus universalis, which he described as follows:

> If one could find characters or signs, apt to express all our thoughts as purely and clearly as arithmetic expresses numbers or analytic geometry expresses lines, one could perform in all subject matters, as far as they are liable to rational thought, what can be done in arithmetic and geometry.[2]
>
> In order to establish a general calculus one has to find characters for arbitrary terms, by which, once they have been connected, the truth of the sentences composed by these expressions can be realized. I found that numbers are the most convenient characters. They are easy to work with and adapt themselves to all subjects, furthermore, they provide certainty.[3]

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[1] Leibniz: „Sämtliche Schriften und Briefe“, Akademie-Ausgabe, 6. Reihe, 4. Band, Teil A, Berlin 1999, which we shall quote as **A**. The important manuscripts on the subject of the present paper are N. 56 to N. 64. We will also refer to the following translations: FS (N. 56 - N. 61, N. 63), P (N. 57, N. 63) and AG (N. 60, N. 62 and N. 64).

[2] **A**, N. 1, „La vraie methode“, p. 6; FS, p.90.

This quotation reveals that Leibniz is the progenitor of some of the main fantasies of our computer age! But this is not the subject here. Instead we will rather examine a particular method which Leibniz invented hoping that it would give him the key to replacing rational reasoning by algorithms based on numbers.

In the context of his arithmetization of Aristotelian logic, Leibniz employs an intensional approach[4] which means that his characteristic numbers stand for terms, denoting concepts (and not for sets of individuals, as in extensional logics). He considers the following types of propositions (propositiones) between terms (we are going to use Leibniz' notation[5]):

U.A.	propositio Universalis Affirmativa, („All x are y", Axy)
P.A.	propositio Particularis Affirmativa, („Some x are y", Ixy)
P.N.	propositio Particularis Negativa, („Some x are not y", Oxy)
U.N.	propositio Universalis Negativa, ("No x are y", Exy).

Leibniz is interested in the construction of a concrete model of Aristotelian logic which consists of numbers. These numbers, belonging to a fixed domain, are substitutes of the abstract terminal symbols x, y, ...On this domain, four concrete arithmetical relations **a, i, o**, and **e** will assume the role of the logical symbols A, I, O, and I of Aristotelian logic, respectively[6].

The main question is: Which domain of numbers – together with which quadruple of relations – qualifies as a model for Aristotelian logic? After answering this question in Section 2, we shall be able to understand and classify Leibniz' different systems of characteristic numbers in Section 3

In his nine manuscripts on the subject of characteristic numbers, Leibniz does not deliver a recipe for the construction of suitable models but simply presents three which differ with respect to the underlying domain of numbers and the definition of the "Aristotelian relations".

Thus, within his first two models, he utilizes the domain of natural numbers, and he chooses the usual relation of divisibility as interpretation of the U.A.- propositio:

> U.A.: Si Propositio Universalis Affirmativa est vera, necesse est ut numerus subjecti dividi possit exacte seu sine residuo, per numerum praedicati.[7]

The idea of this definition is that the relation of divisibility between numbers mirrors the relation between genus and species[8]: Genus (the predicate of a proposition) corresponds to the part, while species (the subject) corresponds to the whole. Accordingly, to the "higher" concept there belongs the smaller number which divides the larger number corresponding to the "lower" concept (the whole) [9]. Here we recognise clearly that Leibniz was aiming at a pure intensional calculus!

The remaining three propositiones are also reduced to the divisibility relation – we shall specify this in Section 3.

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[3] **A**, N. 59, p. 217; FS, p. 203.

[4] For a detailed discussion of the intensional and extensional aspects of Leibniz' logic see Kauppi 1960. Leibniz himself made some unequivocal remarks that he designed his characteristic numbers in the framework of an intensional interpretation of Aristotelian logic. In other parts of his works on logic he uses extensional concepts as well.

[5] The four different propositiones correspond to the Aristotelian predications which are classically denoted by A, I, O, and E.

[6] The relevancy of the notation A / **a,** I **/ i,** O **/ o, and** E **/ e** will be made clear in Section 2.

[7] **A,** N. 56, S. 182. „It is necessary for the universal positive proposition to be true, that the subject number can be divided exactly without remainder by the predicate number".

[8] **A**, N. 57, p. 199 –200.

[9] `„... ita ut generis notio sit pars, speciei notio sit totum, componitur enim ex genere et differentia."` (**A**, N. 57, 11, p. 199).

In order to illustrate his idea, Leibniz, in his first paper on this subject[10], presents the following example:

> Exempli causa, si fingeretur terminus animalis exprimi per numerum aliquem 2 (vel generalis a) terminus rationalis per numerum 3 (vel generalis r) terminus hominus exprimetur per numerum 2 3, id est 6, seu productum ex multiplicatis in vicem 2 et 3 (vel generalius per numerum a r).[11]

The first two models which we shall denote by A1 and A2 respectively, have as underlying domain the set of natural numbers[12]; they differ in their respective definitions of the P.A. – and U.N.- propositiones. Leibniz' third and last model (often called the model of characteristic numbers) will be denoted by Model B in this paper. Its underlying domain is a certain set of pairs of numbers[13].

In this way we will obtain a first rough classification of the three number models by means of the type of the underlying number domain. However, there is a finer classification by fundamental properties of the four "Aristotelian relations". We shall present three criteria to be fulfilled by the relations **a, i, o,** and **e** in order to qualify as Aristotelian relations. These criteria mirror the following three well known properties of classical Aristotelian logic:

Criterion 1:	validity of the central classical syllogism, Barbara.
Criterion 2 :	formalization of Aristotle's method of reasoning by ecthesis.
Criterion 3:	contradiction, a main part of the square of oppositions.

We will also discuss another method of justification of these three criteria, referring to modern model theory of Aristotelian logic[14]. This is not surprising, because Leibniz' manuscripts deal with quite concrete models of Aristotelian logic, and it is just model theory which allows us to examine the relation between models (semantics) and syntax of a logical theory. In Section 2, we will show that the three criteria mentioned above are quite natural assumptions in the framwork of model theory[15].

Both approaches to Leibniz' system lead to the same classification scheme[16] which we resume in Table 1:

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[10] N 56, 17, p. 182.

[11] „If, for example, we assume that the item ‚animal' is expressed by means of the number 2 (or, in general, by a), and the item ‚rational' by means of the number 3 (or, in general, by r), then ‚man' is expressed by 2·3, i.e. 6, as the result of the product of 2 by 3 (or, in general, by the number a·r)".

[12] Here we do not include zero in the set of natural numbers.

[13] In Section 3 we shall see that we could as well say: The underlying domain of model B consists of the set of all rational numbers. As neither Leibniz nor any of his later commentators was aware of or mentioned this observation, we continue to talk about number pairs. Later we shall elucidate the connection of Leibniz' Model B with the set of rational numbers.

[14] This modern theory of Aristotelian logic by means of systems of natural deduction – not being based on predicate calculus - was founded by Corcoran, 1973 and, independently, by Smiley, 1973.

[15] There exist alternatives to our way of proceeding, in particular with respect to our emphasis on ecthesis (e.g. Lukasiewicz ,1951). But it is not our aim to discuss all possible kinds of formalization of Aristotelian logic, but just to develop a special set of instruments for the purpose of understanding and classifying Leibniz' research on characteristic numbers.

[16] Readers who are not interested in modern model theory may skip the somewhat formal Section 2 without missing a central point of our classification and interpretation of Leibniz' number models.

Name of model	Reference to Leibniz' papers	Basic number domain	Criteria satisfied?
A1	N. 56	natural numbers	no
A2	N. 57, 58, 59	natural numbers	yes
B	N. 60 - 64	pairs of natural numbers	yes

Table 1

This table does not show which of the three criteria is not satisfied in Model A1, neither does it reveal why Leibniz enhanced Model A2 in spite of its compliance to all criteria. In Section 3 we will see that though A2 is formally qualified as a model of Aristotelian logic it is not comprehensive enough to be of any practical use.

By means of this paper we aim to disclose the exact formal structure of Leibniz' three number models and also to elucidate the precise relation of these models to each other[17]. We shall see that A1 and A2 differ fundamentally in that only A2 complies with the "canonical" property of ecthesis which links the universal positive (**a**) to the particular positive proposition **i**.

Aristotle employs ecthesis as a method of proof for some syllogisms (Baroco, Bocardo[18]) as well as for the proof of the **e** - conversion[19]. One can also use ecthesis as an alternative to a proof by reductio ad absurdum in certain formal reconstructions of Aristotelian logic[20].

In Model A2, the **a** – relation and the **i** – relation are connected via ecthesis (contrary to Modell A1). Therefore A2 fulfills all conditions of a model in the sense of modern model theory of Aristotelian logic[21]; we will present the details in Section 2.

Concerning model B we will show that Criteria 1 to 3 imply that only one of the four propositiones, the U.A.- propositio, can be chosen independently whilst the others are subordinated. This is of some advantage, as the U.A. – proposition, representing exactly Leibniz' idea of employing divisibility, is the simplest one[22].

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[17] At first sight N. 61 seems to contain an additional „transitional model" of type A2/B, where Leibniz took the U.A.- and P.A. propositiones from Model A2, and the remaining propositiones from Model B. However, this is not possible by pure formal reasons, as the basic number domains of A2 and B are different (cf. Table 1). The whole matter is clarified by commentaries in the Akademie edition, p. 228 and p. 233.

[18] An. Pr. 30a6-14.

[19] „Now, if A belongs to none of the Bs, then neither will B belong to any of the As. For if it does belong to some (for instance to C), it will not be true that A belongs to none of the Bs, since C is one of the Bs." (An. Pr. 25a15-19; transl. by R. Smith, 1989). Here C denotes the term constructed by exposition (ecthesis) which appears neither in the premises nor in the conclusion of the proposition. – Since its invention by Aristotle, this method led to considerable confusion which arose mainly regarding the question of „ontological status" of the „exposed" term C: Does C denote an individual or a concept? Meanwhile, there are diverse proper formalizations of this method (cf. Smith, 1982). – Burkhardt, 1980 writes: „Die Ekthese als Beweis eines Syllogismus findet sich bei Leibniz nicht (Ecthesis as means of proof for syllogisms cannot be found in Leibniz' works)". But we shall see that Leibniz explicitly refers to ecthesis in his construction of the P.A. propositio, even he does not mention the name of the method.

[20] Robin Smith, 1982.

[21] Leibniz described A1 as his first model but abandoned it in favour of A2 from N.57 on without further comment.

[22] It is easy to see that within Model B the definition of the U.A. – propositio generates the definition of the P.N. – propositio by negation. In the same manner the definitions of the P.A. – propositio and the U.N. – propositio are related to each other. This choice of Leibniz conforms to the classical approach (c.f. our Criterion 3). However, our observation that one can even derive the P.A.-propositio (**i**) from the definition of the U.A.-propositio (**a**) is new.

We shall also attempt to make clear that Leibniz’ notation +s-fo r his number pairs in model B is not a good choice as it induces misleading associations with pairs of positive and negative numbers. In opposition to that we will show in Section 4 that the correct way of interpretating the characteristic numbers of model B is to regard them as positive rational numbers s/ .

It is well known that Leibniz ceased to work on the subject of characteristic numbers after having written the manuscript **A,** N. 64. As far as I know, he did not give reasons for abandoning this project. Whereas it has sometimes been said that the system is faulty[23], we know on the contrary, since the work of Lukasiewicz[24], that model B is, in a certain sense, even perfect: It is a model in which exactly[25] the syllogistic deductions of classical Aristotelian logic hold true.

Thus there exist no internal formal errors which are responsible for Leibniz’ abandoning of his efforts on the project of characteristic numbers[26]. But there is evidence that Leibniz, trying to incorporate negative concepts into his formalism reached a point of research where, as we shall see, there was no chance for him to succeed. We also show that, in his last lines concerning this subject, he may even have got muddled by his own plus-minus-notation, struggling hard with a task which he could not resolve within his system. This may indeed have been the reason for his abrupt stop in working on the characteristic numbers – we don’t know. In the last part of our paper we formulate an open question regarding this point of including “negative concepts” into the system of characteristic numbers.

Finally, we will point to an important open problem in the context of the theory of characteristic numbers: Leibniz never attacked the question of how to assign numbers to terms. While he presented some very small examples, he did not even mention the fact that one would need an algorithm for the computation of characteristic numbers in order to realize his dream of replacing thinking by computing. We will formulate this basic problem hoping to stimulate further research on this subject of the “Gödelization of Aristotelian logic”!

In the following Section 2 we give a short introduction into the basics of model theory applied to Aristotelian logic. Its purpose is to show why just the three criteria presented above are the important ones for any model of Aristotelian logic. The reader who is not interested in abstract model theory may skip the whole section without risk of not understanding the other parts of the present paper.

Further content of the paper:

Section 2: Models of Aristotelian Logic

Section 3: Two models of type A

Section 4: Characteristic number pairs

Section 5: Two open questions

Section 6: Final remarks

Section 7: Appendix (Proof of Theorems)

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[23] This view is due to Couturat, 1903 and since then has cut the surface from time to time. Thiel, 1980, correctly pointed out that this view is unfounded; cf. Henrich, 2002, concerning the history of reception of Leibniz’ characteristic numbers.

[24] Lukasiewicz, 1951.

[25] The complicated part is to prove that, within B, there exist no additional valid syllogisms compared to Aristotelian logic; this is a deep result of Slupecki, a pupil of Lukasiewicz.

[26] Therefore also the work of Sotirov, 1999, leads in the wrong direction: He „corrects“ model A2 with the result that, in his model, only finitely many numbers are of importance. This was not Leibniz’ aim who constructed only systems with infinitely many numbers.

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